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Analytical GMD Calculation for Inductions of Rectangular Conductors

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Abstract

In this paper, the theoretical basis and development of geometrical mean distance (GMD) to determine the inductance formula for straight and parallel conducts of rectangular cross section are reviewed. Sample calculations for the self and mutual inductance of dual- and triple-geometric configurations are given by using the proposed formula of GMD theorem.

Keywords: geometrical mean distance, GMD, inductance, mutual inductance.

1. Introduction

The improvements of semiconductor process technologies have enabled the design of on chip inductors with multilevel interconnects [1-3]. The pre-computed tables reported in Refs. [4 and 5] can be employed to calculate the geometrical mean distance (GMD) of two rectangular cross sections with finite thickness. Self- and mutual-inductance effects are importance considerations in the design of electrical circuits because energy storage in magnetic fields can produce voltage transients resulting in noise, feedback, and other undesirable phenomenon. The problem of calculating low-frequency current is solely one of geometry. For conductors carrying high-frequency current, the additional complication of non-uniform current of distribution requires that consideration be given to the skin effects. In either case, expeditious solution of the complex equations is possible only with the aid of computers.

The purpose of this report is to review the theoretical basis of the inductance formula for straight, parallel conductors of rectangular section, illustrating the role of Maxwell’s GMD theorem. Detailed examples are presented for the single conductor and for arrangements of two and three conductors. The general inductance formula and applicable GMD equations have been used with excellent results for applications involving printed circuit cables. The inductance formula is accurate for the range of frequencies where skin depth exceeds conductor thickness.

2. Development of Inductance Equations for Straight and Parallel Conductors

Consider a conductor element in a circuit path, supplied by a source, in which it is desired to establish a current I_1. As the current rises from zero it induces an electromotive force opposing the current rise and this requires the source to supply energy if the current is to be maintained against the induced electromotive force (*emf*). The power expanded in forcing the current against the induce *emf* $e = -L(\mathrm{d}I/\mathrm{d}t)$ is $P = LI(\mathrm{d}I/\mathrm{d}t)$. The total energy supplied in raising the current to the final value I_1 in the time interval t is

$$W_1 = \int_0^t LI\frac{dI}{dt}dt = \int_0^{I_1} LIdI = \frac{1}{2}LI_1^2. \qquad (1)$$

This energy is stored in the magnetic field and is available to the circuit when the source is disconnected.

Consider next a conductor element in each of two circuits supplied by separate sources. If, while current I_1 is being established in circuit 1, a current $I_2^{'}$ is maintained in circuit 2, which is linked by magnetic field interaction with circuit 1, then during the rise of I_1 an *emf* $e = -L_{12}(\mathrm{d}I/\mathrm{dt})$ is induced in circuit 2. The energy required to force the current $I_2^{'}$ against this *emf* is equal to

$$W_2 = \int_0^t \left(L_{12}\frac{dI}{dt}\right)I_2^{'}dt = \int_0^{I_1} L_{12}I_2^{'}dI = L_{12}I_2^{'}I_1 \quad (2)$$

The energy of the system of circuit 1 and 2 is calculated by allowing I_1 in circuit 1 to be established first. The current in circuit 2 is then allowed to rise to I_2 while I_1 is constant. The rise of current in circuit 1 from zero to I_1 involves the storage of energy $(L_1I_1^2)/2$ in the magnetic field. As I_2 is established, energy $(L_2I_2^2)/2$ is supplied by the source 2 while source 1 supplies energy $(L_{12}I_2I_1)/2$ to maintain a constant current I_1. The total energy of the system circuit 1 and circuit 2 can be expressed as

$$W_T = \frac{1}{2}L_1I_1^2 - L_{12}I_2I_1 + \frac{1}{2}L_2I_2^2. \qquad (3)$$

The sign of the second term is negative if the induced *emf* is in such a direction that it aids the flow of current I_2 and supplements the energy of source 2. The parameter of L_{12} is termed the mutual inductance M and defined as the additional energy available, or required, from the vector addition of the two magnetic fields [6]. A mutual inductance of one henry gives rise to an induced *emf* of one voltage when the rate of change of the inducing current 1 A/s. If the *emf* induced in circuit 1 by a current changing at rate of 1 A/s in circuit 2 is equal to *e*, the same *emf e* is induced in circuit 2 when current in circuit 1 changes at the rate 1 A/s. The mutual inductance can also be regarded as the number of flux linkages with circuit 1 due to unit current in circuit 2. The reverse is also true. Self-inductance is simply a special case of mutual inductance, which will shortly become apparent.

The magnetic flux ϕ, needed in Eq.(1) to evaluate the inductance coefficients, can be obtained from the magnetic vector potential $\boldsymbol{A}$ where $\phi = \int \boldsymbol{A} \cdot d\boldsymbol{r}$ and $\boldsymbol{r}$ is the vector of a spherical radius . The equation $\boldsymbol{B}$ = curl $\boldsymbol{A}$ relates the vector potential $\boldsymbol{A}$ and a magnetic field $\boldsymbol{B}$. In the case of a straight rod of circular cross section (radius = a) carrying uniformly distributed current ($\boldsymbol{J}$ of magnitude $\boldsymbol{I}/(\pi a^2)$) and with permeability of free space μ_0, the use of the equation curl $\boldsymbol{B}$ $=\mu_0\boldsymbol{J}$ results in the circumferential magnetic flux density (external to the rod)

$$B_{o\phi} = \frac{\mu_0 I}{2\pi r}, \ r > a. \tag{4}$$

For this geometry, the vector potential has only a component parallel to the current flow. The equation $\boldsymbol{B}$ = curl $\boldsymbol{A}$ reduce to $B_\phi = -dA_z/dr$. Integration of this yields

$$A_{oz} = C - \frac{\mu_0 I}{2\pi}\ln r, \ . \tag{5}$$

where C is constant of integration, inside the rod

$$B_{i\phi} = \frac{\mu_0 I r}{2\pi a^2}, \ r < a \tag{6}$$

and

$$A_{iz} = C - \frac{\mu_0 I r^2}{4\pi a^2}. \tag{7}$$

By requiring the vector potential be continuous at the periphery of the rod, and by arbitrarily assigning a value of zero to the magnitude of vector potentials inside (A_{iz}) and outside (A_{oz}) become

$$A_{iz} = \frac{\mu_0 I}{4\pi a^2}\left(a^2 - r^2\right) \tag{8}$$

and

$$A_{oz} = -\frac{\mu_0 I}{2\pi}\ln\frac{r}{a}, \tag{9}$$

For the case of the field of uniformly distributed current in a straight conductor of any section of area S, the section may be divided into elements ds, each carrying current Ids/S. Equation (5) may be written

$$A = C - \frac{\mu_0}{2\pi}\sum\left(\frac{Ids}{S}\right)\ln r \tag{10}$$

where r is the distance between each element ds and any point P where A is to be calculated. Equation (10) may also be written as

$$A = C - \frac{\mu_0 I}{2\pi S}\int \ln r ds. \tag{11}$$

Computation is greatly facilitated if the section S is divided into small equal elements (ds = S/n), n in number, and if the distance from each element to the point P is measured as $r_1, r_2 \ldots r_n$. Therefore

$$A = C - \frac{\mu_0 I}{2\pi S}\sum_{i=1}^{n}\left(\ln r_i \frac{S}{n}\right)$$

$$= C - \frac{\mu_0 I}{2\pi}\left(\frac{\ln r_1 + \ln r_2 + \ldots + \ln r_n}{n}\right). \tag{12}$$

Let R equal the geometric mean of $r_1, r_2, \ldots r_n$. Then $r_1 r_2 \cdots r_n = R^n$ and

$$\ln r_1 + \ln r_2 + \ldots + \ln r_n = \ln(r_1 r_2 \ldots r_n)$$
$$= \ln(R^n) = n \ln R \tag{13}$$

Equation (12) may now be rewritten as

$$A = C - \frac{\mu_0 I}{2\pi}\ln R. \tag{14}$$

If the n subdivisions are infinitely small, R can be defined by the equation

$$\ln R = \frac{1}{S}\int \ln r dS. \quad (15)$$

Since R is the geometric mean of all possible r distance, it is called the GMD between the point P and the conductor area S. Utilizing Eq. (1) and these forms for the vector potential, Gray [7] writes Eq. (3) in the form

$$W_T = I^2\left(\frac{2}{S_1S_2}\int_{B_2}\int_{B_1}\ln r_{12}dS_1dS_2\right) + I^2\left(-\frac{1}{S_1^2}\int_{B_1}\int_{B_1}\ln r'_{11}dS_1dS'_1\right) + I^2\left(-\frac{1}{S_2^2}\int_{B_2}\int_{B_2}\ln r'_{22}dS_2dS'_2\right). \quad (16)$$

where B_1 and B_2 are two long straight parallel wires of any form of cross section carrying current in opposing directions, r_{12} is the distance between a filament in each of wires 1 and 2, r'_{11} is the distance between two filaments in wire 1 (and correspondingly for r'_{22} in wire 2), and dS'_1 and dS'_2 are differential areas in wires 1 and 2. The permeability in cgs units of the wires and intervening space are taken as unity.

The following expressions can be written:

$$\ln R_{12} = \frac{1}{S_1S_2}\int_{B_2}\int_{B_1}\ln r_{12}dS_1dS_2$$
$$\ln R_{11} = \frac{1}{S_1^2}\int_{B_1}\int_{B_1}\ln r'_{11}dS_1dS'_1 \quad (17)$$
$$\ln R_{22} = \frac{1}{S_2^2}\int_{B_2}\int_{B_2}\ln r'_{22}dS_2dS'_2.$$

Since R_{12}, R_{11}, and R_{22} are the geometric mean distances, R_{12} of area S_1 from area S_2, R_{11} of area S_1 from itself, and R_{22} of area S_2 from itself. The logarithm of each of these distances is termed the logarithmic mean distance. From Eq. (3),the bracket portion of Eq. (16) can be set equal to $L/2$ and the general inductance formula of Snow [8] results for conductors of length ℓ, assuming uniform current distribution in both conductors comprising the return circuit,

$$\frac{L}{\ell} = -\frac{1}{S_1^2}\int_{B_1}\int_{B_1}\ln r'_{11}dS_1dS'_1 - \frac{1}{S_2^2}\int_{B_2}\int_{B_2}\ln r'_{22}dS_2dS'_2 + \frac{4}{S_1S_2}\int_{B_2}\int_{B_1}\ln r_{12}dS_1dS_2. \quad (18)$$

The first two terms are referred to as L_1 and L_2, and called the self inductance of conductors 1 and 2. The last term is called the mutual inductance of conductors 1 and 2. Snow also writes total inductance as

$$\frac{L}{\ell} = \frac{L_1}{\ell} + \frac{L_2}{\ell} - 2\frac{M_{12}}{\ell}, \quad (19)$$

and

$$\frac{L}{\ell} = 4\ln R_{12} - \ln R_{11} - \ln R_{22}. \quad (20)$$

Consequently, the determination of circuit inductance for straight, parallel conductors of any section carrying uniformly distributed current in a closed path depends only on the GMD of the cross-sectional areas from themselves and one another.

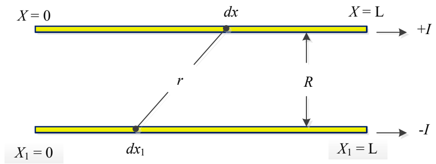

Fig. 1 Current-Carrying filaments.

Neumann's integral [9] can be used to calculate the mutual inductance between the two parallel wire filaments shown in Fig. 1. The integral is developed from Biot and Savart's law for the magnetic field intensity produced at an external point by a current I in an element ds of a circuit. The integral is

$$M = \iint \frac{ds_1 ds_2 \cos\varepsilon}{r} . \tag{21}$$

After the substitution of $ds_1 = dx_1$, $ds_2 = dx$, $\cos\varepsilon = 1$ and $r = \sqrt{R^2 + (x - x_1)^2}$, the integral becomes

$$M = \int_0^\ell dx_1 \int_0^\ell \frac{dx}{\sqrt{R^2 + (x - x_1)^2}} . \tag{22}$$

Curtis [9] shows the details of integration, which results in the mutual inductance formula

$$M = 2\ell\left[\ln\left(\frac{\sqrt{\ell^2 + R^2}}{R}\right) - \frac{\sqrt{\ell^2 + R^2}}{\ell} + \frac{R}{\ell}\right] . \tag{23}$$

When $\ell >>$ R, this equation is frequently simplified as

$$M \approx 2\ell\left[\ln\left(\frac{2\ell}{R}\right) + \frac{R}{\ell} - 1\right] . \tag{24}$$

This is an expression in cgs units. The MKS equivalent is

$$M = 2\times 10^2 \ell\left[\ln\left(\frac{2\ell}{R}\right) + \frac{R}{\ell} - 1\right] . \tag{25}$$

When the medium between the filaments has a relative permeability of one (H will be in nH).

When the separation of two conductors of arbitrary cross section is large relative to the size of their relative cross-sectional dimensions, the mutual inductance of the combination will be essentially the same as that of two filaments along their axes. Typically, however, the cross sections will be too large to justify filament substitution. Each conductor must be divided into an infinite number of filaments and integration will accomplish an average of all possible pair combinations. The change to the basic formula is limited to R, which becomes a GMD function as opposed to a simple centerline distance. Eq. (25) still applies, but R will be replaced by a formidable expression best solved independently. The mutual inductance of the two conductions will be equal to that of two filaments separated by a distance corresponding to the GMD of the two cross sections.

As is evident in Eq. (18), the self inductance of a conductor of any cross section is equal to the sum of the mutual inductances of all filament pairs of the section. As before, the self inductance of a conductor of any section is equal to the mutual inductance of two filaments separated by a distance corresponding to the GMD of the cross section from itself. For this reason, self inductance is often described as a special case of mutual inductance.

The calculation of inductance, then, is indeed a problem of geometry. Equation (25) can be expressed as follows:

$$L \text{ or } M = 2\times 10^{2}\ell\left[\ln\left(\frac{2\ell}{R}\right)+\frac{R}{\ell}-1\right]. \qquad (26)$$

The expression for the total inductance of two parallel nonmagnetic conductors (1 and 2) carrying uniformly distributed current in opposing directions is

$$L_T = L_1 + L_2 - M_{12} - M_{21}. \qquad (27)$$

When the conductors are the same size and shape, Eq. (27) simplified to

$$L_T = 2L_1 - 2M_{12}. \qquad (28)$$

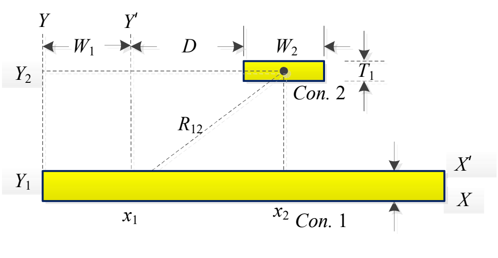

Figure 2. General arrangement of rectangular conductors.

3. GMD of Rectangular Conductors

Higgins [10,11] has derived the logarithmic mean distance formula for two rectangles arbitrarily located in a quadrant, both by the multiple integration method and by the use of complex variables. The multiple-integral equation solved by the two techniques is

$$(W_1T_1W_2T_2)\ln R_{12} = \int_0^{T_1}\int_0^{W_1}\int_{T+T_1}^{T+T_1+T_2}\int_{W_1+D}^{W_1+D+W_2} \ln\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}\,dx_2dy_2dx_1dy_1, \qquad (29)$$

where dx_1dy_1 and dx_2dy_2 are differential areas in the respective conductors. The conductor arrangement is shown in Fig. 2. Higgins' solution for the logarithmic mean distance, which is the logarithm of the GMD R_{12}, is

$$(W_1T_1W_2T_2)\ln R_{12} = -\frac{25}{12}(W_1T_1W_2T_2) - \frac{1}{24}\sum_{i=1}^{4}\sum_{j=1}^{4}(-1)^{i+j}k(A_i,B_j), \qquad (30)$$

where

$$k(A_i,B_j) = \left(A_i^4 - 6A_i^2B_j^2 + B_j^4\right)\ln\sqrt{A_i^2+B_j^2} - A_i^3B_j^1\tan^{-1}\left(\frac{B_j}{A_i}\right) + A_i^1B\tan^{-1}\left(\frac{B_j}{A_i}\right), \qquad (31)$$

$$A_1 = |W_1+D+W_2|;\ A_2 = |D+W_2|;\ A_1 = |D|;\ A_2 = |W_1+D| \qquad (32)$$

and

$$B_1 = |T_1+T+T_2|;\ B_2 = |T+T_2|;\ B_1 = |T|;\ B_2 = |T_1+T|. \qquad (33)$$

The purpose of the second coordinate system (x', y') in figure 2 is to indicate that D and T are both positive only when conductors 1 and 2 are offset as shown. The two values locate only the left and bottom edges of conductor 2 with respect to the right and top edges of conductor 1. For the characteristic over/under orientation of equal-size rectangular conductors, $D = -W_1$ and T is some positive value. For the edge-to-edge orientation of equal-size conductors, $T = T_1$ and D is some positive value.

The logarithmic mean distance of a single rectangle relative to itself can be easily found from the above equations by letting $W_1 = W_2$, $T_1 = T_2$, $D = -W_1$, and $T = -T_1$. Accordingly, $A_1 = A_3 = W_1$, $A_2 = A_4 = 0$, $B_1 = B_3 = T_1$, and $B_2 = B_4 = 0$. The result is

$$\log R_{11} = -\frac{25}{12}\frac{1}{6W_1^2T_1^2}\Big[\big(W_1^4 - 6W_1^2T_1^2 + T_1^4\big) \ln\sqrt{W_1^2 + T_1^2} - 4W_1^3T_1\tan^{-1}\frac{T_1}{W_1} - 4W_1T_1^3\tan^{-1}\frac{W_1}{T_1} - 4W_1^4\ln W_1 - 4T_1^4\ln T_1\Big]. \quad (34)$$

A simple but accurate approximation for the GMD of a single rectangle is $R_{11} = 0.2235(W_1 + T_1)$.

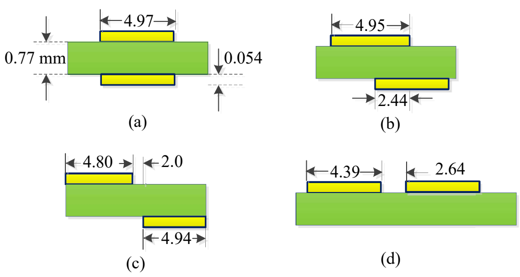

Figure 3. Dual-conductor arrangements.

Table 1 Calculated vs. measured inductance

Case	GMD (mm)	Cal. (nH)	Measured (nH)
(a)	1.693	16.4	16.8
(b)	2.442	30.9	29.7
(c)	4.697	56.8	54.8
(d)	7.244	73.0	72.5

4. Inductance Calculations for Two Parallel Rectangular Conductors

Figure 3 shows in cross section an arrangement of four simple return circuits, each 100 mm long, which were fabricated for the purpose of comparing measures and calculated for the purpose of comparing measures and calculated inductances. All conductors were 0.034 mm (0.0014 inch)-thick copper and were printed on 0.770 mm (0.032 inch)-thick Fiberglass-reinforced epoxy. Table 1 lists the GMD for each conductor pair, calculated for Eq. (30), the total inductance, calculated from Eqs. (26) and (27), and the total inductance measured on a 100 kHz Impedance meter. Agreement between calculated and measured inductance is within 4%. When Eq. (19) is used to calculate the inductance, the values for cases (a) through (d) are 16.6, 31.4, 58.2 and 75.4 nH, respectively. While agreement with the measured values is still satisfactory, more accurate results can generally be obtained with Eq. (25). Experience has shown that inductance values calculate from Eq. (6) for various arrangements of two-conductor return circuits of rectangular cross section agree with measured values within 10%. The inductance and GMD equations have been used principally for circuit less than 1 m long, conductors less than 15 mm wide for dielectric less than 1.57 mm thick.

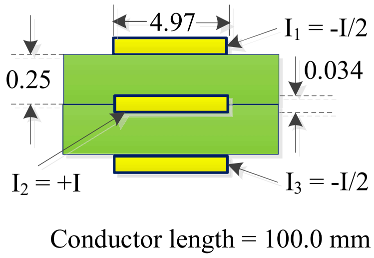

Figure 4. Three-conductor circuit.

5. Inductance Calculations for Three Rectangular Conductors

The instantaneous energy storage in a network of n loops is given by Chen [12] as

$$W_T = \frac{1}{2}L_T I^2 = \frac{1}{2}\sum_{k=1}^{n}\sum_{h}^{n} L_{kh} i_k i_h \, . \qquad (35)$$

For the three-conductor circuit shown in Fig. 4, n = 3. When Eq. (35) is solved for total inductance L_T, the expression is

$$L_T = \frac{1}{4}(L_1 + L_3) + L_2 + \frac{1}{2}M_{13} - M_{12} - M_{23}. \qquad (36)$$

Note that when subscripts k and h are unequal, M_{kh} (for mutual inductance) is used in place of L_{kh}. Eq. (26), (27), (30),and (36) were used to calculate the following results:

$GMD_{1,2,3}$ = 2.239 mm

$GMD_{1,3}$ = 2.630 mm L_T = 9.1 nH

GMD_{12} = GMD_{23} = 2.430 mm

The inductance of many different triple-conductor arrangements can be calculated by this method. As another example, if the center conductor in Fig. 4 is half as wide, but a symmetrical arrangement is retained, the following calculated values are obtained:

$GMD_{1,3}$ = 2.239 mm

GMD_2 = 1.124 mm L_T = 16.9 nH

GMD_{12} = 2095 mm

GMD_{13} = 2095 mm

Eq. (36) and applicable GMD equations give consistently good results for shielded flat cables, and agreement with measured values of inductance is usually within 10%. The equations can also be used to calculate the inductance of the more unusual case of a nonsymmetrical arrangement of three conductors.

6. Conclusions

Accurate analytical formulas for obtaining the GMD of multilevel conductors for the Greenhouse method have been presented. The mutual inductances of multilevel conductors calculated using the proposed formulas. Proposed expressions are scalable and e fficien calculating mutual inductances of interconnects for 2D inductors in VLSI circuits.

References

[1] H. Y. D. Yang, "Design considerations of differential inductors in COMS technology for RFIC", *IEEE RFIC Symp. Dig.*, TX, pp. 449–452, June 2004.

[2] A. Zolfaghari, A. Chan, and B. Razavi, "Stacked inductors and trans- formers in CMOS technology," *IEEE J. Solid-State Circuits*, vol. 36, no. 4, pp. 620–628, April 2001.

[3] P. Park, C. S. Kim, M. Y. Park, S. D. Kim, and H. K. Yu, "Variable inductance multilayer inductor with MOSFET switch control," *IEEE Electron Device Lett.*, vol. 25, no. 3, pp. 144–146, March 2004.

[4] F. W. Grover, "The calculation of the inductance of single-layer coils and spirals wound with wire of large cross section,," *Proc. the Institute of Radio Engineers*, vol. 17, no.11, pp. 2053–2063, Nov. 1929

[5] F. W. Grover, *Inductance Calculations*, Van Nostrand, pp. 1-5, New York, 1946.

[6] J. M. Ham and G. R. Slemn, *Scientific Basic of Electric Engineer*, Van Nostrand, New York, 1946.

[7] A. GRAY, 'Formulars for Computing Capacitance and Inductance', C. Snow, *Absolute Measurements in Electricity and Magnetism*, NBS Circular 544, pp. 40-41, 1954.

[8] H. L. Curtis, *Electricity Measurements*, McGraw-Hill Book Co., Inc., pp. 138-139, New York, 1937.

[9] T. J. Higgins, 'Formulars for the Geometric Mean Distance of Rectangular Areas and of of line Segments', Journal of Physics, N.Y., vol. 14, pp. 188-195, 1943.

[10] T. J. Higgins, 'Theory and Application of Complex logarithmic and Geometric Mean Distances', AIEE Transactions, vol. 66, pp. 12-16, 1947.

[11] W. H. Chen, *Linear Network Design and Sythesis*, McGraw-Hill Book Co., Inc., pp. 57, New York, 1964.

適用於長方形導體電感之解析 GMD 計算

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摘要

本文評述幾何平均距離 (GMD)之理論基礎和推演來確定直線和平行矩形截面導體的電感公式。利用 GMD 定理公式提供自感和互感之二線和三線的導體幾何配置範例計算。

關鍵詞：幾何平均距離, GMD、電感、互感.