# 以半正定規劃演算法設計可調分數延遲 FIR 數位濾波器之新方法

# A New Method for Design of Variable Fractional-Delay (VFD) FIR Digital Filters by Semidefinite Programming (SDP)

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**摘要**

近年來，在可調分數延遲 FIR 數位濾波器之設計大部份會以加權最小平方差逼近法將目標誤差最小化而求得最佳解。但是，最近有一些論文以半正定規劃演算法(SDP)嘗試設計各種的數位濾波器。因此，在本文中，我們將提出以半正定規劃演算法設計可調分數延遲 FIR 數位濾波器，並以實際例子來模擬及證實其效能和可行性。

**關鍵字**：可調分數延遲濾波器，FIR 數位濾波器，可調濾波器，半正定規劃演算法（SDP）。

**Abstract**

For the past decade, the variable fractional-delay FIR digital filters is generally designed by weighted least-squares method to minimize the objective error function and find the optimal solution. Recently, semidefinite programming (SDP) has been found useful in designing various types of FIR digital filters. Therefore, in this paper, the design of variable fractional-delay (VFD) FIR digital filters by semidefinite programming are proposed. A design example will be presented to demonstrate the effectiveness of the proposed method.

**Keywords**：variable fractional-delay filters, FIR digital filters, variable filter, semidefinite programming (SDP).

## 1. Introduction

Recently, the design of variable fractional-delay (VFD) digital filters receives considerable attention due to their wide applications in signal processing and communication systems [1]–[6]. The VFD digital filter is a branch of variable digital filters which are applied to where the frequency characteristics need to be adjustable. Generally, the system is implemented by the famous Farrow structure [1], [5] where a parameter is used to change continuously the delay online without redesigning a new filter. In the paper, the design of VFD FIR digital filters is investigated again by an efficient method. Conventionally, the most works concerning the design of VFD filters are focused on the minimization of the objective error by least-squares approach. The optimization methodology adopted in our proposed method is semidefinite programming (SDP). SDP is primarily concerned with minimizing a linear or convex quadratic objective function subject to linear matrix inequality (LMI) type constraints that depend on the design variables affinely. Recently, it has been deployed in various digital filter design [7]-[12]. Moreover, deriving the elements of relevant matrices in the proposed method, there is no need of numerical integration or closed-form formulation and they can be specified just by the given specifications. Comparing with the existing method, the proposed method doesn't lead to misleading problem and it can't get the elements of relevant matrices due to out of memory. To demonstrate the effectiveness of the proposed method, a design example is presented.

## 2. Problem Formulation and Numerical Example

For designing a VFD FIR filter, the desired response is given by

$$\begin{aligned} D(\omega,p) &= e^{-j\frac{N}{2}\omega}\cdot e^{-j\omega p} \\ &= e^{-j\frac{N}{2}\omega}\left[\cos(\omega p) - j\sin(\omega p)\right], \\ &-\omega_p \le \omega \le \omega_p, \quad -0.5 \le p \le 0.5 \end{aligned} \tag{1}$$

where $N$ and $p$ are the order and the variable fractional-delay of the designed FIR digital filter, respectively. To approximate the desired response in (1), the used transfer function is characterized by

$$H(z,p) = \sum_{n=0}^{N} h_p(n) z^{-n} \tag{2}$$

where the coefficients $h_p(n)$ are expressed as the polynomials of $p$

$$h_p(n) = \sum_{m=0}^{M} h(n,m)\, p^m, \tag{3}$$

hence Eq. (2) becomes

$$\begin{aligned} H(z,p) &= \sum_{n=0}^{N}\sum_{m=0}^{M} h(n,m)\, p^m z^{-n} \\ &= \sum_{m=0}^{M}\left(\sum_{n=0}^{N} h(n,m) z^{-n}\right) p^m \end{aligned} \tag{4}$$

which can be implemented by the famous Farrow structure [1]. In the paper, only even $N$ is considered, and the case for odd $N$ can be developed in a similar manner. By the symmetric/ antisymmetric properties of (1) with respective to $p$, Eq.(4) can be divided into two sub-functions as follows,

$$\begin{aligned} H(z,p) &= \sum_{m=0}^{M_c}\left(\sum_{n=0}^{N} h(n,2m) z^{-n}\right) p^{2m} \\ &+ \sum_{m=1}^{M_s}\left(\sum_{n=0}^{N} h(n,2m-1) z^{-n}\right) p^{2m-1} \end{aligned} \tag{5}$$

where

$$\begin{cases} M_c = M_s = \dfrac{M}{2} & \text{for even } M, \\ M_c + 1 = M_s = \dfrac{M+1}{2} & \text{for odd } M. \end{cases} \tag{6}$$

Also, according to the frequency characteristics of (1), the frequency response of the designed system can be formulated into

$$\begin{aligned} H(e^{j\omega},p) = e^{-j\frac{N}{2}\omega}\Bigg[ &\sum_{m=0}^{M_c}\sum_{n=0}^{N/2} a(n,m) p^{2m}\cos(n\omega) \\ &+ j\sum_{m=1}^{M_s}\sum_{n=1}^{N/2} b(n,m) p^{2m-1}\sin(n\omega)\Bigg] \end{aligned} \tag{7}$$

where

$$a(n,m) = \begin{cases} h\left(\frac{N}{2}, 2m\right), \\ n=0, \quad 0 \le m \le M_c, \\ 2h\left(\frac{N}{2}-n, 2m\right) = 2h\left(\frac{N}{2}+n, 2m\right), \\ 1 \le n \le \frac{N}{2}, \quad 0 \le m \le M_c, \end{cases} \tag{8a}$$

$$b(n,m)=2h\left(\tfrac{N}{2}-n,2m-1\right)$$
$$=-2h\left(\tfrac{N}{2}+n,2m-1\right), \tag{8b}$$
$$1\le n\le \tfrac{N}{2},\quad 1\le m\le M_s$$

Furthermore, substituting $p=0$ into (1) and (7),

$$D(\omega,0)=e^{-j\frac{N}{2}\omega} \tag{9a}$$

and

$$H\left(e^{j\omega},0\right)=e^{-j\frac{N}{2}\omega}\sum_{n=0}^{N/2}a(n,0)\cos(n\omega), \tag{9b}$$

which lead to

$$a(n,0)=\delta(n). \tag{10}$$

Hence, Eq.(7) can be represented by [8]

$$H\left(e^{j\omega},p\right)=e^{-j\frac{N}{2}\omega}\left[1+\sum_{m=1}^{M_c}\sum_{n=0}^{N/2}a(n,m)p^{2m}\cos(n\omega)+j\sum_{m=1}^{M_s}\sum_{n=1}^{N/2}b(n,m)p^{2m-1}\sin(n\omega)\right]. \tag{11}$$

Let **A** and **B** be $\left(\tfrac{N}{2}+1\right)\times M_c$ and $\tfrac{N}{2}\times M_s$ matrices defined by

$$\mathbf{A}=\left[a(n,m),\, 0\le n\le \tfrac{N}{2},\ \ 1\le m\le M_c\right] \tag{12a}$$

and

$$\mathbf{B}=\left[b(n,m),\, 1\le n\le \tfrac{N}{2},\ \ 1\le m\le M_s\right], \tag{12b}$$

respectively. In an variable fractional-delay design, a discretized version of minimizing the weighted $L_2$ error function can be formulated as follow:

$$\underset{\mathbf{A},\mathbf{B}}{\text{minimize}}\ \ \delta \tag{13a}$$

$$\text{subject to}\ \ e(\mathbf{A},\mathbf{B}) \tag{13b}$$

$$e(\mathbf{A},\mathbf{B})=\sum_{i=0}^{K_\omega}\sum_{l=0}^{K_p}W(\omega_i)\left|D(\omega_i,p_l)-H\left(e^{j\omega_i},p_l\right)\right|^2$$
$$=e(\mathbf{A})+e(\mathbf{B}),\quad \omega_i=\frac{i\omega_p}{K_\omega},\quad p_l=\frac{0.5l}{K_p} \tag{13c}$$

where

$$e(\mathbf{A})=\sum_{i=0}^{K_\omega}\sum_{l=0}^{K_p}W(\omega_i)\cdot\left[\cos(\omega_i p_l)-1-\sum_{m=1}^{M_c}\sum_{n=0}^{N/2}a(n,m)\,p_l^{2m}\cos(n\omega_i)\right]^2, \tag{14a}$$

$$e(\mathbf{B})=\sum_{i=0}^{K_\omega}\sum_{l=0}^{K_p}W(\omega_i)\left[-\sin(\omega_i p_l)-\sum_{m=1}^{M_s}\sum_{n=1}^{N/2}b(n,m)\,p_l^{2m-1}\sin(n\omega_i)\right]^2, \tag{14b}$$

a $(K_\omega+1)\times(K_p+1)$ grid is chosen for the error evaluation and $W(\omega)$ is a positive weighting function. In the paper, $K_\omega=200$ and $K_p=60$ are used. It is especially noted that the weighting function depends only on the transform-domain parameter $\omega$, not on the tunable variable $p$. Eq.(14) can be expressed in matrix forms as

$$e(\mathbf{A})=tr\left[\left(\mathbf{D_A}-\mathbf{CAP_A}\right)^T\left(\mathbf{D_A}-\mathbf{CAP_A}\right)\right]$$
$$=tr\left[\mathbf{D_A}^T\mathbf{D_A}-\mathbf{D_A}^T\mathbf{CAP_A}-\left(\mathbf{CAP_A}\right)^T\mathbf{D_A}+\left(\mathbf{CAP_A}\right)^T\left(\mathbf{CAP_A}\right)\right] \tag{15a}$$

and

$$e(\mathbf{B})=tr\left[\left(\mathbf{D_B}-\mathbf{SBP_B}\right)^T\left(\mathbf{D_B}-\mathbf{SBP_B}\right)\right]$$
$$=tr\left[\mathbf{D_B}^T\mathbf{D_B}-\mathbf{D_B}^T\mathbf{SBP_B}-\left(\mathbf{SBP_B}\right)^T\mathbf{D_B}+\left(\mathbf{SBP_B}\right)^T\left(\mathbf{SBP_B}\right)\right] \tag{15b}$$

where $tr(\cdot)$ denotes a trace operator, the superscript $^T$ denotes a transpose operator,

$$\mathbf{D_A}=\left[W^{\frac{1}{2}}(\omega_i)\left(\cos(\omega_i p_l)-1\right),\ 0\le i\le K_\omega,\ 0\le l\le K_p\right], \tag{16a}$$

$$\mathbf{D_B}=\left[-W^{\frac{1}{2}}(\omega_i)\sin(\omega_i p_l),\ 0\le i\le K_\omega,\ 0\le l\le K_p\right], \tag{16b}$$

$$\mathbf{C}=\left[W^{\frac{1}{2}}(\omega_i)\cos(n\omega_i),\ 0\le i\le K_\omega,\ 0\le n\le \tfrac{N}{2}\right], \tag{16c}$$

$$\mathbf{S}=\left[W^{\frac{1}{2}}(\omega_i)\sin(n\omega_i),\ 0\le i\le K_\omega,\ 1\le n\le \tfrac{N}{2}\right], \tag{16d}$$

$$\mathbf{P_A}=\left[p_l^{2m},\ 0\le l\le K_p,\ 1\le m\le M_c\right] \tag{16e}$$

and

$$\mathbf{P_B}=\left[p_l^{2m-1},\ 0\le l\le K_p,\ 1\le m\le M_s\right]. \tag{16f}$$

From (15a)

$$
\begin{aligned}
e(\mathbf{A}) &= c_0 - \left(vec(\mathbf{CA})\right)^T vec(\mathbf{D_A P_A}) \\
&\quad - \left(\left(vec(\mathbf{CA})\right)^T vec(\mathbf{D_A P_A})\right)^T \\
&\quad + \left(vec(\mathbf{CAP}_A^T)\right)^T vec(\mathbf{CAP}_A^T) \\
&= c_0 - \left(vec(\mathbf{CAI})\right)^T vec(\mathbf{D_A P_A}) \\
&\quad - \left(\left(vec(\mathbf{CAI})\right)^T vec(\mathbf{D_A P_A})\right)^T \\
&\quad + \left((\mathbf{P_A} \otimes \mathbf{C}) vec(\mathbf{A})\right)^T (\mathbf{P_A} \otimes \mathbf{C}) vec(\mathbf{A})
\end{aligned} \tag{17}
$$

where $vec$ denotes a vec operator. The operator that transforms a matrix to a vector. If the $m\times n$ matrix $\mathbf{A}$ has $\mathbf{a}_i$ as its $i$th column, then $vec(\mathbf{A}) = \left[\mathbf{a}_0^T, \mathbf{a}_1^T, \cdots, \mathbf{a}_n^T\right]^T$ is the $mn\times 1$ vector. The $c_0 = tr\left(\mathbf{D}_A^T \mathbf{D}_A\right)$ and using the formulation $tr\left[\mathbf{A}^T\mathbf{B}\right] = \left(vec(\mathbf{A})\right)^T vec(\mathbf{B})$, where **A** and **B** are all matrixs $m\times n$.

$$e(\mathbf{A}) \le \delta_A \tag{18}$$

is equivalent to

$$\Gamma_A = \begin{bmatrix} \begin{pmatrix} \delta_A - c_0 + \left(vec(\mathbf{A})\right)^T r_0 \\ + \left(\left(vec(\mathbf{A})\right)^T r_0\right)^T \end{pmatrix} & \left(vec(\mathbf{A})\right)^T q_0^T \\ q_0 vec(\mathbf{A}) & \mathbf{I} \end{bmatrix} \ge 0 \tag{19}$$

where $r_0 = (\mathbf{I}\otimes\mathbf{C})^T vec(\mathbf{D}_A\mathbf{P}_A)$ , $q_0 = \mathbf{P}_A \otimes \mathbf{C}$ and using the formulation $vec(\mathbf{ABC}) = \left(\mathbf{C}^T \otimes \mathbf{A}\right)\cdot vec(\mathbf{A})$ where $\otimes$ denotes the kronecker product and **I** is identity matrix.

Similarly, from (15b)

$$e(\mathbf{B}) \le \delta_B \tag{20}$$

is equivalent to

$$\Gamma_B = \begin{bmatrix} \begin{pmatrix} \delta_B - c_1 + \left(vec(\mathbf{B})\right)^T r_1 \\ + \left(\left(vec(\mathbf{B})\right)^T r_1\right)^T \end{pmatrix} & \left(vec(\mathbf{B})\right)^T q_1^T \\ q_1 vec(\mathbf{B}) & \mathbf{I} \end{bmatrix} \ge 0 \tag{21}$$

where $r_1 = (\mathbf{I}\otimes\mathbf{S})^T vec(\mathbf{D}_B\mathbf{P}_B)$, $q_1 = \mathbf{P}_B \otimes \mathbf{S}$.

Then (13) can be formulaed as

$$\text{minimize } \mathbf{f}^T\mathbf{x} \tag{22a}$$

$$\text{subject to } F(\mathbf{x}) \tag{22b}$$

where $\mathbf{f}^T = [1\ 1\ \mathbf{0}_0\ \mathbf{0}_1]$, $\mathbf{x} = \left[\delta_A\ \delta_B\ \mathbf{A}^T\ \mathbf{B}^T\right]^T$ and $F(\mathbf{x}) = \text{diag}\{\Gamma_A, \Gamma_B\}$. The $\mathbf{0}_0, \mathbf{0}_1$ are the zeros of the length $1\times\left(\frac{N}{2}+1\right)\cdot(M+1)$ and $1\times\frac{N}{2}\cdot(M+1)$, respectively. Clearly, $F(\mathbf{x})$ in (22) is affine w.r.t. $\mathbf{x}$, therefore (22) is a SDP problem. The LMI Control Toolbox from MathWorks Inc. [13] can be used to solve the SDP problem as formulated in (22).

Example : Consider the design of a VFD FIR digital filter with the orders $N$=30, $M$=4, the cutoff frequency $\omega_p = 0.9\pi$, $K_\omega = 200$ and $K_p = 60$.

Fig. 1 presents the group-delay response while the absolute errors of group-delay response $\varepsilon_\tau = 0.008606693925397$. Comparing with $\varepsilon_\tau = 0.00773737$ in [6] , it is near in the absolute errors of group-delay response.

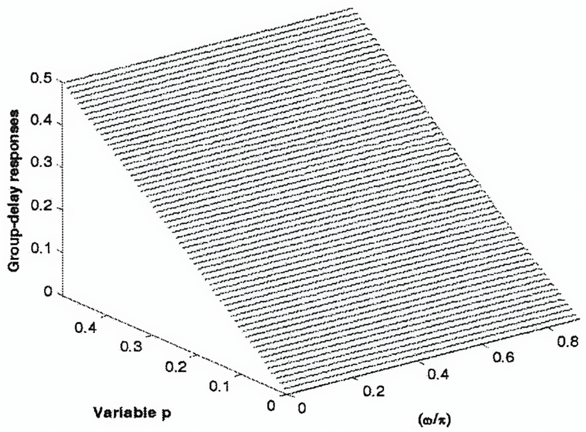


Fig. 1 The group-delay responses of the design of a VFD FIR digital filter by the proposed method.

## 3. Conclusions

In this paper, a new method for design of variable fractional-delay FIR digital filters by Semidefinite Programming is proposed. A design example has been presented to demonstrate the effectiveness. Comparing with the existing method, the proposed method doesn't lead to misleading problem and it can’t get the elements of relevant matrices due to out of memory.

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